

RANDOM SAMPLING OF BANDLIMITED FUNCTIONS

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ABSTRACT. We consider the problem of random sampling for bandlimited functions. When can a bandlimited function f be recovered from randomly chosen samples $f(x_j), j \in J \subset \mathbb{N}$? We estimate the probability that a sampling inequality of the form

$$A\|f\|_2^2 \leq \sum_{j \in J} |f(x_j)|^2 \leq B\|f\|_2^2$$

hold uniformly for all functions $f \in L^2(\mathbb{R}^d)$ with $\text{supp } \hat{f} \subseteq [-1/2, 1/2]^d$ or for some subset of bandlimited functions.

In contrast to discrete models, the space of bandlimited functions is infinite-dimensional and its functions “live” on the unbounded set \mathbb{R}^d . These facts raise new problems and leads to both negative and positive results.

(a) With probability one, the sampling inequality fails for any reasonable definition of a random set on \mathbb{R}^d , e.g., for spatial Poisson processes or uniform distribution over disjoint cubes.

(b) With overwhelming probability, the sampling inequality holds for certain compact subsets of the space of bandlimited functions and for sufficiently large sampling size.

1. INTRODUCTION

The sampling problem asks for the reconstruction or approximation of a function f from its sampled values $\{f(x_j) : j \in J\}$ on some set $\mathcal{X} = \{x_j\} \subseteq \mathbb{R}^d$. In other words, one wants to recover f from given samples $f(x_j)$. This is a many-faceted problem and spreads over many areas of mathematics, engineering, and data processing.

We will impose the standard hypothesis that f is bandlimited. In signal processing this is a realistic assumption, because it amounts to assuming a maximum frequency. The assumption is also relevant in complex analysis because a bandlimited function is just the restriction of an entire function of exponential growth from \mathbb{C}^d to \mathbb{R}^d . The space of bandlimited functions is defined to be

$$\mathcal{B} = \{f \in L^2(\mathbb{R}^d) : \text{supp } \hat{f} \subseteq [-1/2, 1/2]^d\},$$

where we have normalized the spectrum to be the unit cube and the Fourier transform is normalized as $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$.

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The principal goal is to establish a *sampling inequality* of the form

$$(1) \quad A\|f\|_2^2 \leq \sum_j |f(x_j)|^2 \leq B\|f\|_2^2 \quad \forall f \in \mathcal{B}.$$

A set $\{x_j : j \in J\} \subseteq \mathbb{R}^d$ satisfying the sampling inequality (1) is called a set of stable sampling or simply a set of sampling [19]. Once a sampling inequality is established, every $f \in \mathcal{B}$ is uniquely determined by its samples on \mathcal{X} and depends continuously on these samples.

Bandlimited functions in dimension $d = 1$ and $d > 1$ differ in a fundamental way because of the nature of their zeros. In dimension $d = 1$ the zeros of an entire function are always discrete, and there is a precise connection between the possible density of zeros and the growth of f [4, 23, 29]. By contrast, in higher dimensions, the zero sets are analytic manifolds, and standard complex variable techniques do no longer apply. As a consequence, almost everything is known about the sampling of bandlimited functions in dimension $d = 1$, but only a few results are known in higher dimensions, most notably a strong result of Beurling [3].

The difficulties of the sampling of multivariate functions have motivated us to turn to probabilistic techniques and to study *random sampling*. In this approach the sampling set \mathcal{X} is a sequence of random variables $x_j = x_j(\omega)$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in \mathbb{R}^d . The sampling inequality (1) defines an event on Ω , and the goal is to estimate the probability that a random set is a set of sampling.

This point of view has worked successfully in our previous work [1] where we have studied the random sampling of multivariate trigonometric polynomials. We were able to show that some popular numerical algorithms [28] work with “overwhelming” probability. In a similar spirit, Candès, Romberg, and Tao [6, 7] have recently investigated sparse trigonometric polynomials and their reconstruction from a few random samples. The more general context of mathematical learning theory has been studied by Cucker, Poggio, Smale, and Zhou [11, 27, 34]. In [34] sampling in general reproducing kernel Hilbert spaces was studied under the assumption of “rich data.” This amounts to assuming the validity of a sampling inequality. By contrast, our interest is to establish the probability that this basic assumption holds. The common technical point in these approaches [1, 6, 11] is the estimate of entropy and covering numbers and a metric entropy argument.

The first contributions to random sampling of bandlimited functions were perturbation results in dimension $d = 1$. Seip and Ulanovsky [30] investigated random perturbations of regular sampling $\{j + \delta_j : j \in \mathbb{Z}\}$, where δ_j is a sequence of i.i.d. random variables. Chistyakov, Lyubarskii, Pastur [9, 10] studied the more general problem of perturbation of arbitrary Riesz bases of exponentials. These contributions are based on the precise characterization of sampling sets in dimension $d = 1$ [29], and the proofs proceed by estimating the probability that a deterministic condition is satisfied.

For random sampling of bandlimited functions of several variables new types of problems arise.

(a) One cannot fall back on deterministic results in higher dimensions, because sampling theory is not nearly as developed as in dimension $d = 1$. In fact, this is the very reason why we aim for purely probabilistic results.

(b) The space of bandlimited functions \mathcal{B} is infinite-dimensional — in contrast to trigonometric polynomials of given degree or sparsity. Thus random matrix techniques as used in [15, 25] are not applicable.

(c) The configuration space \mathbb{R}^d is non-compact and unbounded — again in contrast to trigonometric polynomials that “live” on the torus $[0, 1]^d$. This raises the question of how to model a sequence of random points in \mathbb{R}^d . On a compact set of positive (Lebesgue) measure the natural notion is that of an independent identically distributed (i.i.d.) sequence of points with uniform distribution. On \mathbb{R}^d there are several natural choices. We will consider two such choices: uniform distributions on disjoint cubes and spatial Poisson processes.

We will prove that for these two concepts of “randomly distributed points on \mathbb{R}^d ” the sampling inequality (1) must fail almost surely (Propositions 2.2 and 2.3). These results come as a surprise to the analyst, but are perhaps more natural for the probabilist. The reasons for the failure of a sampling inequality are either the zeros of entire functions or large holes in the sampling set. In the model of uniform distribution over disjoint cubes, many samples may be near the zeros of a bandlimited function with positive probability. In other words, the lower bound in (1) is small.

In the other model (spatial Poisson process) we show that, with positive probability, there are large holes in the sampling set, which again implies a small lower bound in (1).

To obtain insight into the formulation of positive results, we argue in a practical manner. Realistically one can sample f only on a bounded set; furthermore, every bandlimited function vanishes at infinity, thus samples far out do not contribute anything significant to a sampling inequality. We can learn about $f \in \mathcal{B}$ only if the samples are taken in the “essential support” of f , i.e., the set where most of the L^2 -norm is localized. Thus we will study the subset

$$\mathcal{B}(R, \delta) = \left\{ f \in \mathcal{B} : \int_{[-R/2, R/2]^d} |f(x)|^2 dx \geq (1 - \delta) \|f\|_2^2 \right\}$$

of bandlimited functions. This subset is compact in \mathcal{B} and thus somewhat resembles a finite-dimensional subspace. Since $f \in \mathcal{B}(R, \delta)$ is small outside the cube $[-R/2, R/2]^d$, it should suffice to sample f on the relevant cube. In this way, we are back to a compact configuration space and an almost finite-dimensional function space. Our main result (Theorem 3.1) is a restricted sampling inequality for the subset $\mathcal{B}(R, \delta)$. The proof is a combination of analytic and probabilistic techniques. On the one hand, we will use detailed properties about the spectrum of time-limiting operators on bandlimited functions by Widom [35], on the other hand, the metric entropy method (see e.g., [12]).

The paper is organized as follows. In Section 2 we discuss two natural models for random sequences in \mathbb{R}^d and show that, with probability one, they fail to produce sets of stable sampling. In Section 3 we restrict the attention to a subset

of bandlimited functions and show that on this subset a sampling inequality holds with overwhelming probability. The proof of this result is contained in Section 4. In order to set up the metric entropy method, we discuss the spectrum of time-frequency limiting operators and covering numbers. We mention that we use two distinct inequalities of Bernstein, one from Fourier analysis bounding the L^∞ norm of the derivative, and the other from probability giving estimates for the sums of independent random variables.

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2. NEGATIVE RESULTS

In the case of multivariate trigonometric polynomials, we showed that if one chose points independently and uniformly distributed over the state space, then one could recover the trigonometric polynomial exactly provided only that one had at least as many sample points as the dimension [1, Thm 3.2].

We first show that this is far from the case for bandlimited functions. The difficulty is that the state space is not compact.

We first recall a fundamental necessary condition of Landau for a set of sampling. Let

$$(2) \quad D^-(\mathcal{X}) = \lim_{R \rightarrow \infty} \min_{y \in \mathbb{R}^d} \frac{\text{card } \mathcal{X} \cap (y + [0, R]^d)}{R^d}$$

be the (lower) Beurling density of a set $\mathcal{X} \subseteq \mathbb{R}^d$.

Proposition 2.1. *Assume that $\mathcal{X} = \{x_j\}$ is a set of stable sampling for \mathcal{B} . Then \mathcal{X} must have the following properties:*

(i) $D^-(\mathcal{X}) \geq 1$, in particular there is $R > 0$ such that every cube of side length R contains a sampling point, i.e. $\mathcal{X} \cap (x + [0, R]^d) \neq \emptyset$ for all $x \in \mathbb{R}^d$.

(ii) the number of samples in any cube of length 1 is bounded, $\max_{y \in \mathbb{R}^d} \text{card } \mathcal{X} \cap (y + [0, 1]^d) < \infty$.

A sufficient condition is the following: In dimension $d = 1$, if $D^-(\mathcal{X}) > 1$ and $\inf_{j \neq k} |x_j - x_k| > 0$, then \mathcal{X} is a set of sampling.

Proof. (i) is the result of Landau [19] and have been re-derived in [16] for discrete sampling sets; the general case is an easy extension.

(ii) is an easy consequence of the finiteness of upper bound B in (1).

The sufficient condition in dimension $d = 1$ is usually attributed to Beurling and treated in detail by Seip [29]. \square

Loosely speaking, a set of stable sampling must be dense enough and cannot have arbitrarily large “holes”.

We now consider random sampling sets. Let our probability space be $(\Omega, \mathcal{F}, \mathbb{P})$ and denote points in Ω by ω . When sampling a function f randomly, we consider its samples $f(x_j)$ on a sequence of random points $x_j = x_j(\omega)$. Clearly, a sequence of random points need not have the sufficient density stated in Proposition 2.1. However, if the process is designed to yield only random sets with $D^-(\mathcal{X}) \geq 1$, one

could hope that a generic random set with the necessary density would be a set of stable sampling. This intuition is completely false, as we will show in the next sections.

2.1. Uniform distribution on large disjoint cubes. There are various ways in which one could choose points randomly in \mathbb{R}^d . As a first model we partition \mathbb{R}^d into disjoint cubes $k + [0, 1]^d, k \in \mathbb{Z}^d$, and, in each cube, we choose r points independently and uniformly distributed over $k + [0, 1]^d$. Let \mathcal{X} be the collection of sample points; \mathcal{X} is a random set and thus depends on ω . Clearly $D^-(\mathcal{X}) = r$ almost surely, so one may expect that $\mathcal{X}(\omega)$ is a set of stable sampling with high probability.

Our first result says that one cannot obtain a sampling inequality.

Proposition 2.2. *Let $r \geq 1$ be the number of random samples in each cube $k + [0, 1]^d$. With probability one the following holds:*

For each $k > 0$ there is a function $f_k \in \mathcal{B}$ such that

$$\sum_{x_i \in \mathcal{X}(\omega)} |f_k(x_i)|^2 \leq \frac{1}{k} \|f_k\|_2^2.$$

The function f_k will necessarily depend on ω .

Consequently, a sampling inequality of the form (1) is violated almost surely.

Proof. For notational simplicity we give the proof only in dimension $d = 1$; the case of several variables is treated similarly.

Let

$$g(x) = \frac{\sin(\pi x/2)}{\pi x/2},$$

and let ψ be a nonnegative C^∞ function with support in $[-1/4, 1/4]$ such that $\psi = 1$ on $[-1/8, 1/8]$. Let Ψ be the inverse Fourier transform of ψ and define $F(x) = g(x)\Psi(x)$.

Since ψ and thus Ψ are in the Schwartz class, F is in L^2 , decays rapidly, and there exists a constant c_1 such that $|F(x)| \leq c_1/(1+|x|^2)$. The Fourier transform of F is $\widehat{g} * \psi$, so the support of \widehat{F} lies in $[-1/2, 1/2]$, i.e., $F \in \mathcal{B}$. Since F is bounded, by Bernstein's inequality, F' is also bounded, say, by $c_2 := \|F'\|_\infty$.

Choose N a large even integer so that

$$(3) \quad \sum_{|j| \geq N/2} \frac{c_1^2 r}{(1 + (|j| - 1)^2)^2} < \frac{\|F\|_2^2}{4k}.$$

Choose $\delta > 0$ small so that

$$(4) \quad 2c_2 N r \delta < \frac{\|F\|_2^2}{2k}.$$

Let A_j be the event that in the interval $[j, j+1]$ all r points that were chosen randomly lie within $(j, j+\delta)$ if j is even and within $(j+1-\delta, j+1)$ if j is odd. The events A_j are independent and the probability of A_j is δ^r .

Let $B = \bigcap_{j=-N}^N A_j$ be the event that the samples in $[-N, N]$ are in a δ -neighborhood of the even integers. By independence, the probability of B is $(\delta^r)^{2N+1}$. If $\omega \in B$, then using (3) and our bound on F ,

$$\sum_{x_i \in \mathcal{X}(\omega) \setminus [-N, N]} |F(x_i)|^2 \leq \sum_{|j| \geq N/2} \frac{c_1^2 r}{(1 + (|j| - 1)^2)^2} < \frac{\|F\|_2^2}{4k}.$$

By construction, $F(2j) = 0$ for $j \in \mathbb{Z}$, and so using the bound on F' , we have $|F(x)| \leq c_2 \delta$ if $|x - 2j| \leq \delta$. Therefore if $\omega \in B$, then

$$\sum_{x_i \in \mathcal{X}(\omega) \cap [-N, N]} |F(x_i)|^2 \leq 2c_2 N r \delta < \frac{\|F\|_2^2}{2k}.$$

Combining, if $\omega \in B$, then

$$(5) \quad \sum_{x_i \in \mathcal{X}(\omega)} |F(x_i)|^2 < \frac{\|F\|_2^2}{k}.$$

Now let $C_m = \bigcap_{j=3mN-N}^{3mN+N} A_j$. Clearly the probability of C_m is the same as the probability of B . So $\sum_{m=1}^{\infty} \mathbb{P}(C_m) = \infty$. By independence and the Borel-Cantelli lemma, with probability one, C_m occurs for infinitely many m . If $\omega \in C_m$, let $f_k(x) = F(x - 3mN)$. Clearly, $f_k \in \mathcal{B}$ and the same bounds c_1 and c_2 hold for f_k as for F , provided translation is taken into account. As in (5),

$$\sum_{x_i \in \mathcal{X}(\omega)} |f_k(x_i)|^2 < \frac{\|f_k\|_2^2}{k}.$$

Thus we have proved that, with probability 1, \mathcal{X} fails to be a set of stable sampling. \square

2.2. Spatial Poisson processes. Another scheme of choosing points randomly in \mathbb{R}^d is the spatial Poisson process \mathcal{X} . This means that for some (intensity) function $\lambda : \mathbb{R}^d \rightarrow [0, \infty)$, for any Borel subset of \mathbb{R}^d , the number of points in $\mathcal{X} \cap A$ is a Poisson random variable with parameter $\int_A \lambda(x) dx$. If A_1, \dots, A_n are disjoint sets, then the number of points in $\mathcal{X} \cap A_i$ are independent random variables.

The most natural case is where $\lambda(x)$ is a constant, $\lambda(x) = \rho$, say. Then the expected Beurling density of \mathcal{X} is ρ . Again one might think that \mathcal{X} is a set of stable sampling with high probability. However, as in Proposition 2.2, one cannot get the sampling inequality. In fact, a stronger result is true. One can choose points at a higher rate further from the origin and still have the sampling inequality failing.

Proposition 2.3. *Suppose \mathcal{X} is a spatial Poisson process with $\lambda(x) = o(1 + \log^+(|x|))$. Then with probability one, the sampling inequality (1) fails for every subset \mathcal{Y} of \mathcal{X} .*

Proof. Under the hypothesis on λ , the Beurling density of \mathcal{X} may be infinite. In this case, \mathcal{X} contains too many samples and the upper bound in the sampling inequality (1) will fail to hold. This problem could be fixed by extracting a subset

\mathcal{Y} of \mathcal{X} that satisfies the necessary conditions of Proposition 2.1. So one may still hope that a subsequence \mathcal{Y} may yield a set of stable sampling.

However, we will show that with probability one, for each $k > 0$ there exists a cube of side length k that contains no point of \mathcal{X} . Since the maximal hole of a set of stable sampling is bounded by Proposition 2.1(iii), with probability one, no subset of \mathcal{X} can be a set of stable sampling.

The probability that a Poisson random variable with parameter λ is equal to zero is $e^{-\lambda}$. If X_i , $i = 1, \dots, n$, are independent Poisson variables with parameters λ_i , resp., then

$$\begin{aligned} \mathbb{P}(\text{at least one } X_i \text{ is zero}) &= 1 - \mathbb{P}(X_1 \neq 0, \dots, X_n \neq 0) \\ &= 1 - \prod_{i=1}^n \mathbb{P}(X_i \neq 0) = 1 - \prod_{i=1}^n (1 - e^{-\lambda_i}) \\ &= 1 - \exp\left(\sum_{i=1}^n \log(1 - e^{-\lambda_i})\right) \\ &\geq 1 - \exp\left(-\sum_{i=1}^n e^{-\lambda_i}\right). \end{aligned}$$

Let $\epsilon > 0$ be chosen later. Choose $m_0 > k$ large so that $\lambda(x) \leq \epsilon \log(|x|)$ if $|x| > m_0$. For each $m \geq m_0$ we can find at least m disjoint cubes of side length k lying in $B(0, 3mk) \setminus B(0, 2mk)$; call them C_{m1}, \dots, C_{mm} . The number of points in \mathcal{X} lying in any one of the C_{mj} is a Poisson random variable with parameter less than $c_1 \epsilon (\log m) k^d$. So by the above, the probability that at least one of the C_{mj} is empty is greater than

$$1 - \exp\left(-m e^{-c_1 \epsilon (\log m) k^d}\right).$$

If we choose ϵ so that $c_1 \epsilon k^d \leq 1/2$, then the above probability is greater than

$$1 - \exp\left(-m^{1/2}\right),$$

which will be greater than $1/2$ if m is large enough.

Let D_m be the event that at least one of the cubes C_{mj} , $j = 1, \dots, m$, is empty. For m large, $\mathbb{P}(D_m) \geq 1/2$, and the D_m are independent. So by the Borel-Cantelli lemma the event D_m happens for infinitely many m , with probability 1. In particular, there must be at least one cube of side length k with no points of \mathcal{X} in it. \square

On the other hand, the rate of growth $\log^+(|x|)$ is critical. If the intensity function λ grows faster than some multiple of $\log^+(|x|)$, then the random sequence \mathcal{X} cannot have large holes.

Proposition 2.4. *Suppose \mathcal{X} is a spatial Poisson process with intensity $\lambda(x) \geq c_0(1 + \log^+(|x|))$ for all x . Fix $\alpha > 0$. If $c_0 \geq (d+1)/\alpha^d$, then with probability one, there exists $R > 0$, such that every cube $\alpha k + [0, \alpha]^d$ for $\alpha|k| \geq R$ contains at least one point of \mathcal{X} .*

Proof. Let \mathbb{S}_α be the collection of all cubes of the form $\alpha k + [0, \alpha]^d$, $k \in \mathbb{Z}^d$. We will show that with probability one, all but finitely many cubes in \mathbb{S}_α contain at least one point of \mathcal{X} .

Let C_k be the event that the cube $A = \alpha k + [0, \alpha]^d$ contains no point of \mathcal{X} . If $\alpha|k| \geq N$, then $\lambda(x) \geq c_0 \log N$, and thus $\lambda(A) \geq c_0 \alpha^d \log N$. Thus for $\alpha|k| \geq N$, the probability that this cube is empty is

$$\mathbb{P}(C_k) = e^{-\lambda(A)} \leq e^{-c_0 \alpha^d \log N} = N^{-c_0 \alpha^d}.$$

If we choose $c_0 \alpha^d \geq d + 1$, then $\sum_{k \in \mathbb{Z}^d} \mathbb{P}(C_k) < \infty$. Then by the Borel-Cantelli lemma, the probability that infinitely many of the cubes are empty is 0. Therefore from some R on (depending on ω), all cubes in \mathbb{S}_α that are at least R from the origin are nonempty. \square

3. A POSITIVE RESULT: RELEVANT SAMPLING

The key to the arguments in Section 2 was that random sampling sets have either arbitrarily large holes or can be concentrated near the zeros of a bandlimited function. In the former case we then constructed a class of functions whose main energy is concentrated on the “hole”; in the latter case we constructed a class of functions with prescribed zeros. These classes then violate the sampling inequality.

To obtain positive results we change the focus. Since for no reasonable random sampling set does the norm equivalence (1) hold with positive probability for all bandlimited functions, we will restrict the class of functions for which we ask (1) to hold. The natural idea is to sample a given f in the region where a significant part of the energy is located. In other words, we sample in the region of relevant values.

This idea motivates the following definition. Let $C_R = [-R/2, R/2]^d$ be the cube of length R centered at the origin. Its volume is $\text{vol } C_R = R^d$.

Definition 1. Fix a large number $R > 0$ and a small $\delta \in (0, 1)$. Set

$$(6) \quad \tilde{\mathcal{B}}(R, \delta) = \left\{ f \in \mathcal{B} : \int_{C_R} |f(x)|^2 dx \geq (1 - \delta) \|f\|_2^2 \right\}$$

and

$$(7) \quad \mathcal{B}(R, \delta) = \left\{ f \in \mathcal{B} : \|f\|_2^2 = 1 \text{ and } \int_{C_R} |f(x)|^2 dx \geq 1 - \delta \right\}$$

Then $\mathcal{B}(R, \delta)$ is the subset of \mathcal{B} consisting of those bandlimited functions whose energy is largely concentrated on the cube C_R . Only a fraction δ of the total energy is outside this cube. We note that $\mathcal{B}(R, \delta)$ may be empty when δ is chosen too small. (For an estimate of δ such that $\mathcal{B}(R, \delta) \neq \emptyset$, see Section 3.1) In the following we assume that $\mathcal{B}(R, \delta)$ is non-empty.

It now makes sense to sample such f on the cube C_R and to expect that these samples are relevant and capture the main features of f .

Indeed, we will prove the following result.

Theorem 3.1. *Assume that $\{x_j : j \in \mathbb{N}\}$ is a sequence of i.i.d. random variables that are uniformly distributed over the cube $C_R = [-R/2, R/2]^d$ and $0 < \mu < 1 - \delta$. Then there exist $A, B > 0$ such that the sampling inequality*

$$(8) \quad \frac{r}{R^d}(1 - \delta - \mu)\|f\|_2^2 \leq \sum_{j=1}^r |f(x_j)|^2 \leq \frac{r}{R^d}(1 + \mu)\|f\|_2^2 \quad \forall f \in \tilde{\mathcal{B}}(R, \delta)$$

holds with probability at least

$$1 - 2Ae^{-B\frac{r}{R^d}\frac{\mu^2}{41+\mu}}.$$

The constant B can be taken to be $B = \frac{\sqrt{2}}{36}$. For large R and sufficiently large sampling size r the constant A can be chosen of order $A = \exp(CR^d)$ with C depending on the dimension d .

3.1. Discussion and Open Problems. 1. We emphasize that the exponential probability inequality holds uniformly for all $f \in \tilde{\mathcal{B}}(R, \delta)$. By contrast, for fixed f such an inequality could be derived much more simply from standard limit theorems.

2. Theorem 3.1 is an asymptotic result. It is effective only for sufficiently large sampling sizes. To achieve (8) with a probability exceeding $1 - \epsilon$, we need $2Ae^{-B\frac{r}{R^d}\frac{\mu^2}{41+\mu}} < \epsilon$ or

$$(9) \quad r \geq \frac{R^d(41 + \mu)}{B\mu^2} \left(\log \frac{2}{\epsilon} + CR^d \right) = \mathcal{O}(R^{2d}).$$

Since $\mathcal{B}(R, \delta)$ sits in a space of approximate dimension $D = R^d$, we need $\mathcal{O}(D^2)$ samples to recover every $f \in \mathcal{B}(R, \delta)$. In finite dimensional problems, for instance, when sampling trigonometric polynomials of fixed degree, one can often use random matrix techniques to show that the effective number of samples is in fact of the order $\mathcal{O}(D \log D)$ [15]. It is open whether this bound is achievable for bandlimited functions in $\mathcal{B}(R, \delta)$.

3. The sampling inequality (8) states that every $f \in \mathcal{B}(R, \delta)$ is uniquely determined by a sufficient, but *finite* number of samples in $[-R/2, R/2]^d$. This may seem paradoxical at first glance, because the set of bandlimited functions such that $f(x_j) = 0$ for $j = 1, \dots, r$, is an infinite-dimensional subspace of \mathcal{B} . However, as we assume that f is essentially supported on the cube $[-R/2, R/2]^d$, this means that f must take large values there. If $f(x_j) = 0$ for sufficiently many $x_j \in [-R/2, R/2]^d$, then f would oscillate and thus have a large derivative. But this would contradict the bandlimitedness, which implies that the derivatives of f are bounded by π . While Theorem 3.1 is a probabilistic result, it seems possible to also prove a deterministic sampling inequality (8) for $\mathcal{B}(R, \delta)$, at least in dimension $d = 1$.

4. We emphasize that $\mathcal{B}(R, \delta)$ is not a subspace. This means that the frame algorithm (a linear reconstruction method) [13] cannot be used to recover f from its samples. Likewise, the projection-onto-convex-sets (POCS) method cannot be applied, because $\mathcal{B}(R, \delta)$ is not convex. Although (8) determines each $f \in \mathcal{B}(R, \delta)$ uniquely, currently we do not have an explicit reconstruction algorithm to recover f from its relevant samples.

4. PROOF OF THEOREM 3.1

Theorem 3.1 will be a consequence of a large deviation inequality that holds uniformly over the whole class $\mathcal{B}(R, \delta)$ and will be proved in the following sections.

4.1. Time-Frequency Limiting Operators. Let P_R and Q be the projection operators defined by

$$(10) \quad P_R f = \chi_{C_R} f \quad \text{and} \quad Qf = \mathcal{F}^{-1}(\chi_{[-1/2, 1/2]^d} \hat{f}),$$

where \mathcal{F}^{-1} is the inverse Fourier transform. Then Q is the orthogonal projection from $L^2(\mathbb{R}^d)$ onto \mathcal{B} and P_R is the restriction of a function to the cube C_R . The composition

$$(11) \quad A_R = QP_RQ$$

is the operator of time and frequency limiting. This operator has been studied in detail by Landau, Slepian, Pollak [20, 21, 31–33] and many others. It encodes many deep properties of bandlimited functions and their restrictions. In particular, A_R is a compact positive operator of trace class and a precisely known eigenvalue distribution.

We summarize the properties of the spectrum that will be needed in the sequel.

Let $A_R^{(1)}$ denote the operator of time-frequency limiting in dimension $d = 1$. Explicitly, $A_R^{(1)}$ is defined on $L^2(\mathbb{R})$ by the formula

$$(A_R^{(1)} f)^\wedge(\xi) = \int_{-1/2}^{1/2} \frac{\sin \pi R(\xi - \eta)}{\pi(\xi - \eta)} \hat{f}(\eta) d\eta \quad \text{for } |\xi| \leq 1/2.$$

We denote its eigenvalues by $\mu_k = \mu_k(R)$ in decreasing order and indicate the dependence on R . Then the first $[R]$ eigenvalues are approximately 1, followed by a “plunge region” of thickness $\mathcal{O}(\log R)$ after which the remaining eigenvalues are almost zero. Precisely, $\mu_{[R]+1}(R) \leq 1/2 \leq \mu_{[R]-1}(R)$; see [18]. This behavior of the eigenvalues is usually formulated by saying that functions with spectrum $[-1/2, 1/2]$ and “essential” support on $[-R/2, R/2]$ form a finite-dimensional subspace of “approximate” dimension R . In particular, we may think of $\mathcal{B}(R, \delta)$ as a subset of a finite-dimensional space of dimension R .

The precise asymptotic behavior of the μ_k for $k \rightarrow \infty$ was obtained by Widom [35, Lemmas 1–3]: he showed that for large k

$$(12) \quad \mu_k(R) \asymp 2\pi \left(\frac{\pi R}{8} \right)^{2k+1} \frac{1}{k!^2},$$

where $a_k \asymp b_k$ means that $\lim_{k \rightarrow \infty} a_k/b_k = 1$. In particular, (12) implies the super-exponential decay

$$(13) \quad \mu_k(R) \leq C \exp \left(-2k \log \left(\frac{2k}{\pi R} \right) \right).$$

We will use the following weaker exponential estimate.

Lemma 4.1. [35] *Given $\alpha > 0$ there exists a constant $\kappa > 0$, such that*

$$(14) \quad \mu_k(R) \leq e^{-k/\kappa} \quad \text{for } k \geq \frac{R}{1-\alpha}.$$

REMARK: This result is an asymptotic result for both $R \rightarrow \infty$ and $k \rightarrow \infty$. We emphasize that the constant κ depends only on α , but not on R . (Widom works with the operator $\int_{-1}^1 \frac{\sin \gamma(\xi-\eta)}{\pi(\xi-\eta)} \hat{f}(\eta) d\eta$, so a simple dilation shows that we have to use $\gamma = \pi R/2$ to obtain $A_R^{(1)}.$)

The largest eigenvalue μ_0 of $A_R^{(1)}$ is the operator norm of $A_R^{(1)}$ and is of size $\mu_0(R) = 1 - 2\pi\sqrt{2R}e^{-\pi R}(1 + \mathcal{O}(R^{-1}))$ by a result of Fuchs [14]. Thus up to terms of higher order the operator norm of A_R is $\lambda_0 = \mu_0^d = 1 - 2\pi d\sqrt{2R}e^{-\pi R}$. Assume that $\mathcal{B}(R, \delta) \neq \emptyset$ and $f \in \mathcal{B}(R, \delta)$, then

$$1 - \delta \leq \int_{C_R} |f(t)|^2 dt = \langle A_R f, f \rangle \leq \lambda_0.$$

This implies that $\delta \geq 1 - \lambda_0 \geq 2\pi d\sqrt{2R}e^{-\pi R}$ (up to terms of higher order in R).

Next, let $C(\epsilon)$ be the function counting the number of eigenvalues of $A_R^{(1)}$ exceeding ϵ , precisely

$$(15) \quad C(\epsilon) = \text{card}\{\mu_k : \mu_k \geq \epsilon\}.$$

Then Lemma 4.1 implies that

$$(16) \quad C(\epsilon) \leq \frac{R}{1-\alpha} + \kappa \log \frac{1}{\epsilon}.$$

A different estimate for the eigenvalue count was obtained by Landau and Widom [22]:

$$(17) \quad C(\epsilon) = R + \frac{2}{\pi} \log \frac{1-\epsilon}{\epsilon} \log R + o(\log R).$$

However, this is an asymptotic result for $R \rightarrow \infty$, and its proof leaves open whether the term $o(\log R)$ can be chosen independent of ϵ . By contrast, the weaker estimate (16) works with a constant κ independent of R , at the price of the factor $(1-\alpha)^{-1}$. Since we need the eigenvalue behavior for *fixed* R , we use Widom's earlier result.

Next consider the time-frequency limiting operator A_R on $L^2(\mathbb{R}^d)$. Clearly A_R is the d -fold tensor product of $A_R^{(1)}$, $A_R = A_R^{(1)} \otimes \cdots \otimes A_R^{(1)}$. Consequently, λ is an eigenvalue of A_R , $\lambda \in \sigma(A_R)$, if and only if $\lambda = \prod_{j=1}^d \mu_{k_j}$, where $\mu_{k_j} \in \sigma(A_R^{(1)})$ is an eigenvalue of the one-dimensional operator $A_R^{(1)}$. Since $0 < \mu_k < 1$, we have $\prod_{j=1}^d \mu_{k_j} \geq \epsilon$ only when $\mu_{k_j} \geq \epsilon$ for $j = 1, \dots, d$. Consequently,

$$(18) \quad \{\lambda \in \sigma(A_R) : \lambda \geq \epsilon\} \subseteq \{\lambda = \prod_{j=1}^d \mu_{k_j} : \mu_{k_j} \in \sigma(A_R^{(1)}), \mu_{k_j} \geq \epsilon\}.$$

We arrange the eigenvalues of A_R by magnitude $1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_n \geq \lambda_{n+1} \geq \cdots > 0$ and again let $C(\epsilon) = \max\{n : \lambda_n \geq \epsilon\}$ be the function counting the

number of eigenvalues of A_R exceeding ϵ . We choose $\alpha = 1/2$ and combine (16) and (18); then the eigenvalue distribution for A_R in dimension d is

$$(19) \quad C(\epsilon) \leq \left(2R + \kappa \log \frac{1}{\epsilon}\right)^d,$$

where κ is independent of R and ϵ .

4.2. Covering Number for $\mathcal{B}(R, \delta)$. Recall that the covering numbers $N(\epsilon) = N(C, \epsilon)$ of a compact set C in a Banach space are defined to be the minimum number of balls of radius less than or equal to ϵ required to cover C . For the covering number of balls in Euclidean space we use a well-known estimate, see [8, p. 9] and [11, Prop. 5].

Lemma 4.2. *Let $D(0, r) = \{x \in \mathbb{C}^d : \|x\|_2 \leq r\}$ be the ball of radius r in \mathbb{C}^d . The covering number of $D(0, r)$ is given by*

$$(20) \quad N(\epsilon) = e^{2d \log \frac{4r}{\epsilon}}.$$

Let us note that the covering number of the shell $D(0, r) \setminus D(0, r(1 - \delta))$ for some $\delta > 0$ is of the order $N(\epsilon) = e^{2d \log \frac{4r}{\epsilon}} - e^{2d \log \frac{4r(1-\delta)}{\epsilon}} = e^{2d \log \frac{4r}{\epsilon}} (1 - e^{2d \log(1-\delta)})$. The difference from the covering number of the full ball is thus negligible for large dimensions.

In the main part of the argument we will use the restriction of bandlimited functions to the cube C_R . Therefore we will use the local norms

$$\begin{aligned} \|f\|_{2,R} &= \left(\int_{C_R} |f(x)|^2 dx \right)^{1/2}, \\ \|f\|_{\infty,R} &= \sup_{x \in C_R} |f(x)|, \end{aligned}$$

and we denote the restriction of $\mathcal{B}(R, \delta)$ to C_R by

$$V(R, \delta) = P_R \mathcal{B}(R, \delta) = \{f \in L^2(C_R) : f = \chi_{C_R} h \text{ for } h \in \mathcal{B}(R, \delta)\}.$$

Lemma 4.3. (i) $V(R, \delta)$ is a compact subset in $L^2(C_R)$.

(ii) The covering number $N_2(\epsilon)$ of $V(R, \delta)$ (with respect to $\|\cdot\|_{2,R}$) is bounded by

$$(21) \quad N_2(\epsilon) \leq \exp \left(2^{d+1} \left(R + \kappa \log \frac{2\sqrt{\delta}}{\epsilon} \right)^d \log \frac{4\sqrt{2}}{\epsilon} \right).$$

Proof. The finiteness of the covering numbers implies that $V(R, \delta)$ is compact, so it suffices to prove (ii).

(ii) Let φ_n be the normalized eigenfunctions of A_R corresponding to the eigenvalues of λ_n . (These are tensor products of the standard prolate spheroidal functions.) Then $\{\varphi_n : n \in \mathbb{N}\}$ is an orthonormal basis for \mathcal{B} . If $f = \sum_{n \in \mathbb{N}} c_n \varphi_n \in \mathcal{B}$, then $\|f\|_2^2 = \sum_{n \in \mathbb{N}} |c_n|^2$ and

$$(22) \quad \|f\|_{2,R}^2 = \int_{C_R} |f(x)|^2 dx = \langle A_R f, f \rangle = \sum_{n \in \mathbb{N}} |c_n|^2 \lambda_n.$$

Consequently $f \in \mathcal{B}(R, \delta)$ if and only if $\mathbf{c} \in S_\delta = \{\mathbf{c} \in \ell^2 : \|\mathbf{c}\|_2 = 1, \sum_{n=1}^\infty |c_n|^2 \lambda_n \geq 1 - \delta\}$. Then $V(R, \delta)$ (with the local $\|\cdot\|_{2,R}$ -norm) and S_δ (with the weighted ℓ^2 -norm) are isomorphic and their covering numbers are identical.

We first determine a suitable cutoff D such that the remainder $\sum_{n>D} |c_n|^2 \lambda_n$ is uniformly small over S_δ : since $\lambda_n < 1$, we have

$$\begin{aligned} 1 - \sum_{n>D} |c_n|^2 &\geq \sum_{n \leq D} |c_n|^2 \lambda_n \\ &\geq 1 - \delta - \sum_{n>D} |c_n|^2 \lambda_n \\ &\geq 1 - \delta - \lambda_{D+1} \sum_{n>D} |c_n|^2. \end{aligned}$$

We obtain that

$$\delta \geq (1 - \lambda_{D+1}) \sum_{n>D} |c_n|^2,$$

and thus

$$(23) \quad \sum_{n>D} |c_n|^2 \lambda_n \leq \lambda_{D+1} \sum_{n>D} |c_n|^2 \leq \frac{\lambda_{D+1}}{1 - \lambda_{D+1}} \delta.$$

Given $\epsilon > 0$, we choose the minimal D so that $\lambda_{D+1} < \frac{\epsilon^2}{4\delta}$, then $\frac{\lambda_{D+1}}{1 - \lambda_{D+1}} \delta \leq 2\lambda_{D+1} \delta < \epsilon^2/2$. According to (19) we may choose D to be

$$(24) \quad D = C \left(\frac{\epsilon^2}{4\delta} \right) \leq \left(2R + \kappa \log \frac{4\delta}{\epsilon^2} \right)^d = 2^d \left(R + \kappa \log \frac{2\sqrt{\delta}}{\epsilon} \right)^d.$$

Once D is determined, choose an $\frac{\epsilon}{\sqrt{2}}$ -net $\{\mathbf{a}_j : j = 1, \dots, N_2\}$ for the unit ball in \mathbb{C}^D (with respect to the Euclidean norm) and set $f_j = \sum_{n \leq D} \mathbf{a}_j(n) \varphi_n$. By Lemma 4.2 the cardinality of this net is at most $N_2 = e^{2D \log \frac{4\sqrt{2}}{\epsilon}}$.

Given $f = \sum_{n=1}^\infty c_n \varphi_n \in \mathcal{B}(R, \delta)$, choose \mathbf{a}_j and the corresponding $f_j \in \mathcal{B}$, such that $\sum_{n \leq D} |c_n - \mathbf{a}_j(n)|^2 < \epsilon^2/2$. Then by (23) and the definition of D

$$\begin{aligned} \|f - f_j\|_{2,R}^2 &= \sum_{n \leq D} |c_n - \mathbf{a}_j(n)|^2 \lambda_n + \sum_{n>D} |c_n|^2 \lambda_n \\ &\leq \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2. \end{aligned}$$

Thus $\{f_j\}$ is an ϵ -net for $V(R, \delta)$ with respect to $\|\cdot\|_{2,R}$. Now by Lemma 4.2 and (24) imply that the cardinality of this ϵ -net is at most

$$\begin{aligned} N_2(\epsilon) &\leq \exp \left(2D \log \frac{4\sqrt{2}}{\epsilon} \right) \leq \exp \left(2C \left(\frac{\epsilon^2}{4\delta} \right) \log \frac{4\sqrt{2}}{\epsilon} \right) \\ (25) \quad &\leq \exp \left(2^{d+1} \left(R + \kappa \log \frac{2\sqrt{\delta}}{\epsilon} \right)^d \log \frac{4\sqrt{2}}{\epsilon} \right). \end{aligned}$$

□

REMARK: In dimension $d = 1$ a similar estimate for the covering number was obtained in [17]. Estimate (25) also follows from general principles in approximation theory [26] (Ch. 4.2, in particular Thm. 2.5 and Cor. 2.6). The estimate of the covering number by means of the eigenvalue distribution, equivalently between entropy numbers and approximation numbers, goes back to an inequality of Mityagin [24, Ch. 9].

As our next step we want a similar estimate for the covering number of $V(R, \delta)$ in the local $\|\cdot\|_{\infty, R}$ -norm. For this recall a basic inequality for bandlimited functions: If $f \in \mathcal{B}$, then

$$(26) \quad \|f\|_{\infty} \leq \|f\|_2 \quad \forall f \in \mathcal{B}.$$

A similar comparison for the local norms is given in the next lemma.

Lemma 4.4. *If $f \in \mathcal{B}$, then*

$$(27) \quad \|f\|_{\infty, R} = \max_{x \in C_R} |f(x)| \leq K_d \|f\|_2^{\frac{d}{d+2}} \|f\|_{2, R}^{\frac{2}{d+2}},$$

where the constant K_d depends only on the dimension d and is of order $\mathcal{O}(d)$.

Proof. We assume first that f is real-valued and that $\alpha = \max_{x \in C_R} |f(x)|$ is taken at $x_0 \in C_R$. By switching sign if necessary, we have $\alpha = f(x_0) \geq |f(x)|$ for all $x \in C_R$. Next observe that by (26) and Bernstein's inequality we have

$$\left\| \frac{\partial f}{\partial x_j} \right\|_{\infty} \leq \left\| \frac{\partial f}{\partial x_j} \right\|_2 \leq \pi \|f\|_2 \quad \text{for } f \in \mathcal{B},$$

and consequently

$$\|\nabla f\|_{\infty} = \max_{x \in \mathbb{R}^d} \left(\sum_{j=1}^d \left| \frac{\partial f}{\partial x_j}(x) \right|^2 \right)^{1/2} \leq \pi \sqrt{d} \|f\|_2 \quad \text{for } f \in \mathcal{B}.$$

Since $f(x) = f(x_0) + \nabla f(\xi) \cdot (x - x_0)$ for some $\xi \in \mathbb{R}^d$, we obtain a lower estimate for f near its maximum at x_0 by

$$|f(x)| \geq \alpha - \|\nabla f\|_{\infty} |x - x_0| \geq \alpha - \pi \sqrt{d} \|f\|_2 |x - x_0| \geq 0$$

on the ball $B(x_0, \beta) = \{x : |x - x_0| \leq \alpha/(\pi \sqrt{d} \|f\|_2) := \beta\}$. We note that $\beta = \|f\|_{\infty, R}/(\pi \sqrt{d} \|f\|_2) \leq (\pi \sqrt{d})^{-1}$ by (26), and thus a fixed portion of the ball $B(x_0, \beta)$ is always contained in C_R .

Consequently (with σ_{d-1} denoting the surface of the $d - 1$ -dimensional unit ball in \mathbb{R}^d)

$$\begin{aligned}
\int_{C_R} |f(x)|^2 dx &\geq \int_{C_R \cap B(x_0, \beta)} (\alpha - \pi\sqrt{d}\|f\|_2 |x - x_0|)^2 dx \\
&\geq \frac{1}{2^d} \int_{B(x_0, \beta)} (\alpha - \pi\sqrt{d}\|f\|_2 |x - x_0|)^2 dx \\
&= \frac{1}{2^d} \int_{B(0, \beta)} (\alpha - \pi\sqrt{d}\|f\|_2 |x|)^2 dx \\
&= \frac{1}{2^d} \sigma_{d-1} d\pi^2 \|f\|_2^2 \int_0^\beta (\beta - r)^2 r^{d-1} dr \\
&= \frac{1}{2^d} \sigma_{d-1} d\pi^2 \|f\|_2^2 \frac{2}{d(d+1)(d+2)} \beta^{d+2}
\end{aligned}$$

Unraveling this inequality, we obtain that

$$\begin{aligned}
\|f\|_{\infty, R} &= \max_{x \in C_R} |f(x)| = \pi\sqrt{d}\|f\|_2 \beta \\
&\leq \left(2^{d-1} \sigma_{d-1}^{-1} (d+1)(d+2)\right)^{\frac{1}{d+2}} \left(\pi\sqrt{d}\right)^{\frac{d}{d+2}} \|f\|_2^{\frac{d}{d+2}} \left(\int_{C_R} |f(x)|^2 dx\right)^{\frac{1}{d+2}} \\
&= K'_d \|f\|_2^{\frac{d}{d+2}} \|f\|_{2, R}^{\frac{2}{d+2}}.
\end{aligned}$$

For complex-valued $f \in \mathcal{B}$, we have to take $K_d = 2K'_d = 2(2^{d-1} \sigma_{d-1}^{-1} (d+1)(d+2))^{\frac{1}{d+2}} (\pi\sqrt{d})^{\frac{d}{d+2}}$. Using $\sigma_{d-1} = d\pi^{d/2}/\Gamma(d/2 + 1)$, one can then show that $K_d = \mathcal{O}(d)$. \square

Corollary 4.5. (i) $V(R, \delta)$ is a compact subset in $\mathcal{C}([-R/2, R/2]^d)$.

(ii) The covering number $N(\epsilon)$ of $V(R, \delta)$ with respect to $\|\cdot\|_{\infty, R}$ is bounded by

$$(28) \quad N(\epsilon) \leq \exp \left(2^{d+1} \left(R + \kappa \left(\frac{d}{2} + 1 \right) \log \frac{2K_d}{\epsilon} \right)^d \log \frac{4K_d}{\epsilon} \right).$$

Proof. Given $\epsilon > 0$, set $\epsilon_0 = 2^{-d/2} \left(\frac{\epsilon}{K_d} \right)^{d/2+1}$ and let $\{f_j\}$ by an ϵ_0 -net with respect to $\|\cdot\|_{2, R}$. If $f \in V(R, \delta)$ and $\|f - f_j\|_{2, R} \leq \epsilon_0$, then we have

$$\|f - f_j\|_{\infty, R} \leq K_d \|f - f_j\|_2^{\frac{d}{d+2}} \|f - f_j\|_{2, R}^{\frac{2}{d+2}} \leq K_d 2^{\frac{d}{d+2}} \epsilon_0^{\frac{2}{d+2}} \leq \epsilon.$$

Thus $\{f_j\}$ is an ϵ -net for $V(R, \delta)$ with respect to $\|\cdot\|_{\infty, R}$ and $N(\epsilon) \leq N_2(\epsilon_0)$. Now use Lemma 4.3 and estimate the occurring logarithmic term by $\log \frac{2\sqrt{\delta}}{\epsilon_0} = \log \frac{2\sqrt{\delta} 2^{d/2} K_d^{d/2+1}}{\epsilon^{d/2+1}} \leq \left(\frac{d}{2} + 1 \right) \log \frac{2K_d}{\epsilon}$. \square

The precise order of the covering number for $d = 1$ with respect to the local supremum norm $\|\cdot\|_{\infty, R}$ was derived by Buslaev and Vitushkin [5]. Their technique is specifically one-dimensional and yields $N(\epsilon) = e^{R \log(C/\epsilon)}$ for some constant $C > 0$.

We will work with ϵ -nets in the $\|\cdot\|_{\infty,R}$ -norm for $\epsilon = 2^{-\ell}$, $\ell = 1, 2, \dots$. In this case the covering number can be rewritten as

$$(29) \quad N(2^{-\ell}) \leq \exp \left(2^{d+1} \left(R + \left(\frac{d}{2} + 1 \right) \kappa((\ell+1) \log 2 + \log K_d) \right)^d ((\ell+2) \log 2 + \log K_d) \right) := \exp p(\ell),$$

where $p(\ell) = 2^{d+1} \left(R + \left(\frac{d}{2} + 1 \right) \kappa((\ell+1) \log 2 + \log K_d) \right)^d ((\ell+2) \log 2 + \log K_d)$ is a polynomial of degree $d+1$.

What is crucial in the above estimate, is that the exponent grows polynomially in ℓ , but not faster.

4.3. Preparation for the proof of Theorem 3.1. Assume that $\{x_j : j \in \mathbb{N}\}$ is an infinite sequence of i.i.d. random variables, each of which is uniformly distributed over the cube C_R .

For every $f \in \mathcal{B}$ we introduce the random variable

$$(30) \quad Y_j(f) = |f(x_j)|^2 - \frac{1}{R^d} \int_{C_R} |f(x)|^2 dx = |f(x_j)|^2 - \mathbb{E}[|f(x_j)|^2].$$

Then $Y_j(f)$ is a sequence of independent random variables with $\mathbb{E} Y_j(f) = 0$.

We first estimate the probability distribution of the random variable

$$\sup_{f \in \mathcal{B}(R, \delta)} \sum_{j=1}^r Y_j(f).$$

For the repeated application of Bernstein's inequality for sums of independent random variables we will need the following estimates for the $Y_j(f)$'s.

Lemma 4.6. *Let $f, g \in \mathcal{B}(R, \delta)$ and $j \in \mathbb{N}$. Then the following inequalities hold:*

$$(31) \quad \text{Var } Y_j(f) \leq \frac{1}{R^d},$$

$$(32) \quad \text{Var } (Y_j(f) - Y_j(g)) \leq \frac{4}{R^d} \|f - g\|_{\infty, R}^2,$$

$$(33) \quad \|Y_j(f)\|_{\infty} \leq 1,$$

$$(34) \quad \|Y_j(f) - Y_j(g)\|_{\infty} \leq 2\|f - g\|_{\infty, R}.$$

Proof. We abbreviate the expected value of $|f(x_j)|^2$ by $m(f) = R^{-d} \int_{C_R} |f(x)|^2 dx$. Using (26), we obtain

$$\begin{aligned} \text{Var } Y_j(f) &= \mathbb{E}[Y_j(f)^2] = \mathbb{E}[|f(x_j)|^4] - m(f)^2 \\ &= \frac{1}{R^d} \int_{C_R} |f(x)|^4 dx - m(f)^2 \\ &\leq \frac{1}{R^d} \|f\|_{\infty, R}^2 \|f\|_2^2 \leq \frac{1}{R^d}. \end{aligned}$$

Similarly, we obtain

$$\|Y_j(f)\|_{\infty} = \sup_{\omega \in \Omega} \left| |f(x_j(\omega))|^2 - m(f) \right| \leq \max \left(\|f\|_{\infty, R}^2, \frac{1}{R^d} \int_{C_R} |f(x)|^2 dx \right) \leq 1.$$

To prove (32), we write

$$\begin{aligned}
\text{Var}(Y_j(f) - Y_j(g)) &= \mathbb{E}(Y_j(f) - Y_j(g))^2 \\
&= \frac{1}{R^d} \int_{C_R} (|f(x)|^2 - |g(x)|^2)^2 dx - (m(f) - m(g))^2 \\
&\leq \frac{1}{R^d} \int_{C_R} |f(x) - g(x)|^2 (|f(x)| + |g(x)|)^2 dx \\
&\leq \frac{2}{R^d} \|f - g\|_{\infty, R}^2 \int_{\mathbb{R}^d} (|f(x)|^2 + |g(x)|^2) dx \leq \frac{4}{R^d} \|f - g\|_{\infty, R}^2
\end{aligned}$$

The last estimate follows similarly from

$$\begin{aligned}
\|Y_j(f) - Y_j(g)\|_{\infty} &\leq \sup_{\omega \in \Omega} \left(\left| |f(x_j(\omega))|^2 - |g(x_j(\omega))|^2 \right| - \frac{1}{R^d} \int_{C_R} (|f(x)|^2 - |g(x)|^2) dx \right) \\
&\leq \| |f|^2 - |g|^2 \|_{\infty, R} \\
&\leq \|f - g\|_{\infty, R} \| |f| + |g| \|_{\infty} \\
&= 2\|f - g\|_{\infty, R}.
\end{aligned}$$

□

4.4. Proof of the sampling inequality. The sampling inequality follows from a uniform large deviation inequality for the sampling of bandlimited functions.

Theorem 4.7. *Let $\{x_j : j \in \mathbb{N}\}$ is a sequence of i.i.d. random variables that are uniformly distributed over $C_R = [-R/2, R/2]^d$. Then there exist constants $A, B > 0$ depending on d and R , such that*

$$(35) \quad \mathbb{P} \left(\sup_{f \in \mathcal{B}(R, \delta)} \left| \sum_{j=1}^r Y_j(f) \right| \geq \lambda \right) \leq 2A \exp \left(-B \frac{\lambda^2}{41rR^{-d} + \lambda} \right)$$

for $r \in \mathbb{N}$ and $\lambda \geq 0$.

Here $B = \frac{\sqrt{2}}{36}$. If R is sufficiently large, A is of order $A = \exp(CR^d)$ for a constant depending only on d and κ .

Before we prove the large deviation inequality, we show how the main theorem follows from Theorem 4.7.

Proof of Theorem 3.1. Choose $\lambda = \frac{r\mu}{R^d}$ and recall that $Y_j(f) = |f(x_j)|^2 - R^{-d} \int_{C_R} |f(x)|^2 dx$.

Thus the event $\mathcal{E} = \{\sup_{f \in \mathcal{B}(R, \delta)} |\sum_{j=1}^r Y_j(f)| \leq r\mu R^{-d}\}$ coincides with the event

$$(36) \quad \frac{r}{R^d} \int_{C_R} |f(x)|^2 dx - \frac{r\mu}{R^d} \leq \sum_{j=1}^r |f(x_j)|^2 \leq \frac{r}{R^d} \int_{C_R} |f(x)|^2 dx + \frac{r\mu}{R^d} \quad \text{for all } f \in \mathcal{B}(R, \delta).$$

Since by definition $1 - \delta \leq \int_{C_R} |f(x)|^2 dx \leq 1$, we find that the event of the uniform sampling inequality

$$(37) \quad \frac{r(1 - \mu - \delta)}{R^d} \leq \sum_{j=1}^r |f(x_j)|^2 \leq \frac{r(1 + \mu)}{R^d} \quad \text{for all } f \in \mathcal{B}(R, \delta)$$

is contained in \mathcal{E} . As a consequence of Theorem 4.7 the sampling inequality (37) holds uniformly for all $f \in \mathcal{B}(R, \delta)$ with probability at least

$$1 - 2A \exp(-BrR^{-d}\mu^2/(41 + \mu)).$$

This proves Theorem 3.1. \square

We are left to prove the probability estimate of Theorem 4.7. To estimate the probability of the deviation of a sum of random variables from its average we use Bernstein's inequality for the sums of independent random variables [2]: *Let $Y_j, j = 1, \dots, r$, be a sequence of bounded, independent random variables with $\mathbb{E}Y_j = 0$, $\text{Var}Y_j \leq \sigma^2$, and $\|Y_j\|_\infty \leq M$ for $j = 1, \dots, r$. Then*

$$(38) \quad \mathbb{P}\left(\left|\sum_{j=1}^r Y_j\right| \geq \lambda\right) \leq 2 \exp\left(-\frac{\lambda^2}{2r\sigma^2 + \frac{2}{3}M\lambda}\right).$$

Proof of Theorem 4.7. Step 1: A metric entropy argument. For a given $\ell \in \mathbb{N}$, we construct an $2^{-\ell}$ -covering for $V(R, \delta)$ with respect to the local norm $\|\cdot\|_{\infty, R}$. Let \mathcal{A}_ℓ be the corresponding $2^{-\ell}$ -net for $\ell = 1, 2, \dots$. Then \mathcal{A}_ℓ has cardinality at most $N(2^{-\ell}) \leq e^{p(\ell)}$ for some polynomial of degree $d + 1$ by Corollary 4.5.

Given $f \in \mathcal{B}(R, \delta)$, let f_ℓ be the function in $\mathcal{A}(2^{-\ell})$ that is closest to f in $\|\cdot\|_{\infty, R}$ -norm, with some convention for breaking ties. Since $\|f - f_\ell\|_{\infty, R} \rightarrow 0$ we can write

$$Y_j(f) = Y_j(f_1) + (Y_j(f_2) - Y_j(f_1)) + (Y_j(f_3) - Y_j(f_2)) + \dots.$$

If $\sup_{f \in \mathcal{B}(R, \delta)} |\sum_{j=1}^r Y_j(f)| \geq \lambda$, then \mathcal{E}_ℓ must hold for some $\ell \geq 1$, where

$$\mathcal{E}_1 = \left\{ \text{there exists } f_1 \in \mathcal{A}(1/2) \text{ such that } \left| \sum_{j=1}^r Y_j(f_1) \right| \geq \lambda/2 \right\}$$

and

$$\begin{aligned} \mathcal{E}_\ell = & \left\{ \text{there exist } f_\ell \in \mathcal{A}(2^{-\ell}), \text{ and } f_{\ell-1} \in \mathcal{A}(2^{-\ell+1}) \text{ with} \right. \\ & \|f_\ell - f_{\ell-1}\|_{\infty, R} \leq 3 \cdot 2^{-\ell}, \\ & \left. \text{such that } \left| \sum_{j=1}^r (Y_j(f_\ell) - Y_j(f_{\ell-1})) \right| \geq \lambda/2\ell^2 \right\}. \end{aligned}$$

If this were not the case, then, with $f_0 = 0$,

$$\left| \sum_{j=1}^r Y_j(f) \right| \leq \sum_{\ell=1}^{\infty} \left| \sum_{j=1}^r (Y_j(f_\ell) - Y_j(f_{\ell-1})) \right| \leq \sum_{\ell=1}^{\infty} \frac{\lambda}{2\ell^2} = \frac{\pi^2}{12} \lambda < \lambda.$$

Next we estimate the probability of \mathcal{E}_ℓ .

Step 2. We estimate the term $\ell = 1$ separately. For fixed $f \in \mathcal{A}(1/2)$, the probability of the event \mathcal{E}_1 is bounded, using Bernstein's inequality (38) and Lemma 4.6, by

$$2 \exp\left(-\frac{\lambda^2/4}{2r\text{Var}Y_j(f) + \frac{2}{3}(\lambda/2)\|Y_j(f)\|_\infty}\right) \leq 2 \exp\left(-\frac{\lambda^2}{2rR^{-d} + \lambda/3}\right).$$

There are at most $N(1/2) = \exp\left(2^{d+1}(R + \kappa(\frac{d}{2} + 1) \log 4K_d)^d \log 8K_d\right)$ functions in $\mathcal{A}(1/2)$, so the probability of \mathcal{E}_1 is bounded by

$$(39) \quad 2 \exp\left(2^{d+1}(R + \kappa(\frac{d}{2} + 1) \log 4K_d)^d \log 8K_d\right) \exp\left(-\frac{\lambda^2}{2rR^{-d} + \lambda/3}\right).$$

Step 3. For $\ell \geq 2$, we estimate the probability of \mathcal{E}_ℓ in a similar fashion by using Lemma 4.6, (32), and (34). If $f \in \mathcal{A}(2^{-\ell})$ and $g \in \mathcal{A}(2^{-\ell+1})$ with $\|f - g\|_{\infty, R} \leq 3 \cdot 2^{-\ell}$, we have

$$\begin{aligned} \mathbb{P}\left(\left|\sum_{j=1}^r (Y_j(f) - Y_j(g))\right| > \frac{\lambda}{2\ell^2}\right) \\ \leq 2 \exp\left(-\frac{\lambda^2/4\ell^4}{2r \cdot 4 \cdot R^{-d}(3 \cdot 2^{-\ell})^2 + \frac{2}{3}2 \cdot 3 \cdot 2^{-\ell-1}\lambda/\ell^2}\right) \\ = 2 \exp\left(-\frac{2^\ell}{8\ell^2} \frac{\lambda^2}{36rR^{-d}\ell^2 2^{-\ell} + \lambda}\right). \end{aligned}$$

Note $36\ell^2/2^\ell < 41$. There are at most $N(2^{-\ell})$ functions in $\mathcal{A}(2^{-\ell})$ and $N(2^{-\ell+1})$ functions in $\mathcal{A}(2^{-\ell+1})$. Finally, this can happen for any ℓ . So the probability of $\bigcup_{\ell=2}^{\infty} \mathcal{E}_\ell$ is bounded by

$$(40) \quad \begin{aligned} \sum_{\ell=2}^{\infty} N(2^{-\ell})N(2^{-\ell+1})2 \exp\left(-\frac{2^\ell}{8\ell^2} \frac{\lambda^2}{41rR^{-d} + \lambda}\right) \\ \leq \sum_{\ell=2}^{\infty} 2 \exp\left(p(\ell) + p(\ell-1) - \frac{2^\ell}{8\ell^2} \frac{\lambda^2}{41rR^{-d} + \lambda}\right), \end{aligned}$$

where we use (29) for the covering number.

Step 4. We will need the following inequality:

If $p, a > 0$, then

$$(41) \quad \sum_{\ell=2}^{\infty} e^{-a^\ell p} \leq \frac{1}{pa \log a} e^{-ap}.$$

This inequality follows from the integral test and the substitution $a^x = u$:

$$\begin{aligned} \sum_{\ell=2}^{\infty} e^{-a^\ell p} &\leq \int_1^{\infty} e^{-a^x p} dx \\ &= \frac{1}{\log a} \int_a^{\infty} e^{-pu} \frac{du}{u} \\ &\leq \frac{1}{a \log a} \int_a^{\infty} e^{-pu} du \\ &= \frac{1}{pa \log a} e^{-ap}. \end{aligned}$$

Step 5. To estimate the sum (40), we rewrite and simplify each term. Set

$$(42) \quad \psi = \frac{\lambda^2}{41rR^{-d} + \lambda}$$

$$(43) \quad c_1 = \min_{\ell \geq 2} \frac{2^{\ell/2}}{8\ell^2}$$

$$(44) \quad c_2 = \max_{\ell \geq 2} \frac{2p(\ell)}{2^{\ell/2}}.$$

and $\sup_{\ell \geq 2} \ell^2/2^\ell = 9/8$. Then the ℓ -th term in (40) is majorized by

$$\exp\left(-2^{\ell/2}(c_1\psi - c_2)\right).$$

If $\psi > 0$ is large enough so that $p := c_1\psi - c_2 > 0$, then (41) implies that

$$(45) \quad \begin{aligned} \mathbb{P}\left(\bigcup_{\ell=2}^{\infty} \mathcal{E}_\ell\right) &\leq 2 \frac{1}{(c_1\psi - c_2)\sqrt{2}\log\sqrt{2}} e^{-\sqrt{2}(c_1\psi - c_2)} \\ &= \frac{2\sqrt{2}}{\log 2} \frac{e^{\sqrt{2}c_2}}{c_1\psi - c_2} \exp\left(-\frac{\sqrt{2}c_1\lambda^2}{41rR^{-d} + \lambda}\right). \end{aligned}$$

Since the term for $\ell = 1$ has the same form, we have proved that

$$\mathbb{P}\left(\sup_{f \in \mathcal{B}(R, \delta)} \left|\sum_{j=1}^r Y_j(f)\right| \geq \lambda\right) \leq 2A \exp\left(-\frac{\sqrt{2}c_1\lambda^2}{41rR^{-d} + \lambda}\right),$$

whenever $\psi > c_2/c_1$.

For the exponent B we may take the smaller of the exponents in (39) and (45), i.e., $B = \min(3, \sqrt{2}c_1)$. If we choose λ large enough, so that $c_1\psi - c_2 \geq \frac{2\sqrt{2}}{\log 2}$, then we may take $A = \max(\exp(2^{d+1}(R + \kappa(\frac{d}{2} + 1)\log 4K_d)^d \log 8K_d), e^{\sqrt{2}c_2})$. Thus we have proved Theorem 4.7.

Step 6. To obtain an idea of the magnitude of the constants involved, we give some rough estimates for c_1 and c_2 , A and B .

For c_1 we obtain

$$c_1 = \frac{1}{8} \min_{\ell \geq 2} \frac{2^{\ell/2}}{\ell^2} = \frac{1}{36},$$

so the exponent B in (35) is $\sqrt{2}c_1 = \frac{\sqrt{2}}{36}$, which is approximately ≈ 0.0393 .

As for c_2 , recall that $p(\ell) = 2^{d+1} \left(R + (\frac{d}{2} + 1)\kappa((\ell + 1)\log 2 + \log K_d)\right)^d ((\ell + 2)\log 2 + \log K_d)$. If $(\frac{d}{2} + 1)\kappa((\ell + 1)\log 2 + \log K_d) \leq R$, then

$$\frac{p(\ell)}{2^{\ell/2}} \leq 2^{d+1}(2R)^d \max_{\ell \geq 2} \frac{(\ell + 2)\log 2 + \log K_d}{2^{\ell/2}} \leq c_3 R^d.$$

In the other case, we may estimate $p(\ell)/2^{\ell/2}$ by a constant that depends on d , K_d and κ , but not on R . Thus for R sufficiently large, we obtain $c_2 \leq c_3 R^d$ and $A \leq \exp(CR^d)$.

Finally consider the condition $c_1\psi - c_2 \geq \frac{2^{3/2}}{\log 2}$, which follows from $\frac{\lambda^2}{41rR^{-d}+\lambda} \geq c_4R^d \geq \frac{c_2+2^{3/2}/\log 2}{c_1}$. Since $x \geq B + \sqrt{D}$ implies $x^2 \geq Bx + D$, we find that

$$\lambda \geq c_4 + (41c_4rR^{-d})^{1/2},$$

for a constant independent of R . □

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